

# Symmetries of Bianchi I space-times

Michael Tsamparlis and Pantelis S. Apostolopoulos

Department of Physics, Section of Astrophysics-Astronomy-Mechanics,  
University of Athens, Zografos 15783, Athens, Greece

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## Abstract

All diagonal proper Bianchi I space-times are determined which admit certain important symmetries. It is shown that for Homotheties, Conformal motions and Kinematic Self-Similarities the resulting space-times are defined explicitly in terms of a set of parameters whereas Affine Collineations, Ricci Collineations and Curvature Collineations, if they are admitted, they determine the metric modulo certain algebraic conditions. In all cases the symmetry vectors are explicitly computed. The physical and the geometrical consequences of the results are discussed and a new anisotropic fluid, physically valid solution which admits a proper conformal Killing vector, is given.

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## 1 Introduction

Collineations are geometrical symmetries which are defined by the general relation:

$$\mathcal{L}_\xi \Phi = \Lambda \quad (1)$$

where  $\Phi$  is any of the quantities  $g_{ab}$ ,  $\Gamma_{bc}^a$ ,  $R_{ab}$ ,  $R_{bcd}^a$  and geometric objects constructed by them and  $\Lambda$  is a tensor with the same index symmetries as  $\Phi$ . Some of the well known (and most important) types of collineations are: Conformal Killing vector (CKV)  $\xi^a$  defined by the requirement  $\mathcal{L}_\xi g_{ab} = 2\psi g_{ab}$  and reducing to a Killing vector (KV) when  $\psi = 0$ , to a Homothetic vector field (HVF) when  $\psi = \text{const.}$  and to a Special Conformal Killing vector (SCKV) when  $\psi_{;ab} = 0$ . Proper Affine Conformal vector (ACV) is defined by the requirement  $\mathcal{L}_\xi g_{ab} = 2\psi g_{ab} + 2H_{ab}$  where  $H_{ab;c} = 0$ ,  $\psi_{;ab} \neq 0$  and reducing to an Affine Vector (AV) when  $\psi_{;a} = 0$  and to a Special Affine Conformal vector (SACV) when  $\psi_{;ab} = 0$ . Curvature Collineation (CC) defined by the requirement  $\mathcal{L}_\xi R_{bcd}^a = 0$  and finally Ricci Collineation (RC) defined by the requirement  $\mathcal{L}_\xi R_{ab} = 0$ .

Collineations other than motions (KVs) can be considered as non-Noetherian symmetries and can be associated with constants of motion and, up to the level of CKVs, they can be used to simplify the metric [1]. For example AVs are related to conserved quantities [2] (a result used to integrate the geodesics in FRW space-times), RCs are related to the conservation of

particle number in FRW space-times [3] and the existence of CCs implies conservation laws for null electromagnetic fields [4].

The set of (smooth) collineations of a space-time can be related with an inclusion relation leading to a tree like inclusion diagram [4] which shows their relative hierarchy. A collineation of a given type is proper if it does not belong to any of the subtypes in this diagram. In order to relate a collineation to a particular conservation law and its associated constant(s) of motion the properness of the collineation must first be assured.

A different type of symmetry we shall discuss, which is of a kinematic nature, is the Kinematic Self Similarity (KSS). It is defined by the requirements [5, 6]  $\mathcal{L}_\xi u_a = \alpha u_a$ ,  $\mathcal{L}_\xi h_{ab} = 2\delta h_{ab}$  where  $u^a$  is the four velocity of the fluid,  $h_{ab} = g_{ab} + u_a u_b$  ( $u_a u^a = -1$ ) is the projection tensor normal to  $u^a$  and  $\alpha, \delta$  are constants. A KSS reduces to a HVF when  $\alpha = \delta \neq 0$  and to a KV when  $\alpha = \delta = 0$ . The KSS are characterised by the scale independent ratio  $\alpha/\delta$ , which is known as the similarity index. When  $\alpha = 0$  the KSS is of type zero (zeroth kind) and when  $\delta = 0$  it is of type infinite. Kinematic Self Similarity should be regarded as the relativistic generalisation of self similarity of Newtonian Physics rather than the generalisation of the space-time homotheties. From the physical point of view the detailed study of cosmological models admitting KSS shows that they can represent asymptotic states of more general models or, under certain conditions, they are asymptotic to an exact homothetic solution [7, 8].

A diagonal Bianchi I space-time is a spatially homogeneous space-time which admits an abelian group of isometries  $G_3$ , acting on spacelike hypersurfaces, generated by the spacelike KVs  $\xi_1 = \partial_x, \xi_2 = \partial_y, \xi_3 = \partial_z$ . In synchronous co-ordinates the metric is:

$$ds^2 = -dt^2 + A_\mu^2(t)(dx^\mu)^2 \quad (2)$$

where the metric functions  $A_1(t), A_2(t), A_3(t)$  are functions of the time co-ordinate only (Greek indices take the space values 1, 2, 3 and Latin indices the space-time values 0, 1, 2, 3). When two of the functions  $A_\mu(t)$  are equal (e.g.  $A_2 = A_3$ ) the Bianchi I space-times reduce to the important class of plane symmetric space-times (a special class of the Locally Rotational Symmetric space-times [9, 10]) which admit a  $G_4$  group of isometries acting multiply transitively on the spacelike hypersurfaces of homogeneity generated by the vectors  $\xi_1, \xi_2, \xi_3$  and  $\xi_4 = x^2 \partial_3 - x^3 \partial_2$ . In this paper we are interested only in *proper diagonal* Bianchi I space-times (which in the following will be referred for convenience simply as Bianchi I space-times), hence all metric functions are assumed to be different and the dimension of the group of isometries acting on the spacelike hypersurfaces is three.

A general Bianchi I space-time does not admit a given collineation. The demand that it does, acts like a "selection rule" by selecting those Bianchi I space-times whose metric functions  $A_\mu(t)$  satisfy a certain set of differential equations or algebraic conditions depending on the collineation. These conditions do not necessarily have a solution. For example Coley and Tupper [11] have determined all space-times admitting an ACV. It is easy to check that no (*proper*) Bianchi I space-time belongs to these space-times. In fact it can be shown that the demand that a Bianchi I space-time admits an ACV leads to the conditions  $A_3(t) = \text{const.}$  and  $A_1(t) = A_2(t)$  i.e. the plane symmetric case.

Although Bianchi I space-times are important in the study of anisotropies and they have served as the basis for this study, it appears that their collineations have not been considered in the literature. For example even at the level of proper conformal symmetries, the CKV found by Maartens and Mellin [12] is a CKV in an LRS space-time and not in a Bianchi I

space-time. Perhaps this is due to the fact that the direct solution of the collineation equations is difficult. However there are many general results due to Hall and his co-workers - which will be referred subsequently as they are required - which make possible the determination of the collineations and the corresponding Bianchi I space-times that admit them without solving any difficult differential equations.

In Section II we determine all Bianchi I space-times which admit (proper or not) CKVs. We show that the only Bianchi I space-times which admit a proper HVF are the Kasner-type space-times. In Section III we study the smooth RCs and show that, provided that the Ricci tensor is non-degenerate, there are four families of (proper) Bianchi I space-times admitting smooth RCs. The metrics of these families are determined up to a set of algebraic conditions among the metric functions whereas the corresponding collineation vectors are computed in terms of a set of constant parameters. In Section IV we study the smooth CCs of Bianchi I space-times and show that, assuming the non degeneracy of the Ricci tensor, there are no Bianchi I space-times which admit proper CCs. In Section V we consider the KSS symmetry and determine all Bianchi I space-times which admit a KSS. A particular result which generalises the previous result concerning the HVF, is that the only Bianchi I space-times admitting a proper KSS (of the second kind) are the Kasner-type space-times. In Section VI we discuss the physical implications of these results and examine the compatibility of the physical assumptions on the type of the matter (perfect fluid, electromagnetic field etc.) with the various types of symmetry. An apparently new Bianchi I viscous fluid solution is found, with non zero bulk viscous stress. The model begins with a big-bang and isotropises at late times tending to a Robertson-Walker model (de Sitter Universe).

## 2 Conformal symmetries

To determine the Bianchi I space-times which admit CKVs we use the theorem of Defrise-Carter [13, 14, 15] which has been reconsidered and improved by Hall and Steele [16, 17]. This Theorem concerns the reduction of the conformal algebra  $\mathcal{G}$  of a metric, to the Killing/Homothetic algebra of a (globally defined) conformally related metric. Hall has shown that it is not always possible to find a conformal scaling required by Defrise-Carter, but Hall and Steele showed that the local result can be regained if one imposes at each space-time point the restrictions (a) space-time has the same Petrov type and (b) the dimension of  $\mathcal{G}$  is constant. However, if the Petrov type is I,D,II at a point these restrictions are not necessary. By a direct computation of the Weyl tensor of the general Bianchi I metric (1) we find that the Petrov type is either I (or its degeneracy type D which corresponds to the LRS case which we ignore in this paper) or type O (conformally flat) (in fact all Bianchi type space-times of class A are Petrov type I or its specialisations [18]). This means that we have two cases to consider i.e. conformally flat and non conformally flat space-times.

### 2.1 Bianchi I space-times of Petrov type I

It is known that the maximum dimension of the conformal algebra of a space-times of Petrov type is I or II is four [16]. Hence the Bianchi I space-time admits at most exactly one *proper* CKV  $\mathbf{Y}$  (the properness is assured provided that  $Y^0 \neq 0$ ) and consequently the four dimensional

conformal algebra  $\mathcal{G}_4 = \{\xi_1, \xi_2, \xi_3, \mathbf{Y}\}$ . From the Defrise-Carter theorem it follows that there exists a smooth function  $U(x^a)$  such that  $\mathcal{G}_4$  restricts to a Lie algebra of KVs for the metric  $d\hat{s}^2 = U^2(x^a)ds^2$ . Because the vectors  $\{\xi_1, \xi_2, \xi_3\}$  are KVs for both metrics we deduce that  $U(x^a) = U(t)$  and the type of space-time (i.e. Bianchi I) is retained. Hence the problem of determining the CKVs of Bianchi I space-times is reduced to the determination of the extra KV.

Assuming  $\mathbf{Y} = \mathbf{Y}^\tau(x^i)\partial_\tau + Y^\mu(x^i)\partial_\mu$  where  $\tau = \int U(t)dt$  Jacobi identities and Killing equations imply the relations:

$$\mathbf{Y} = \partial_\tau + ax\partial_x + by\partial_y + cz\partial_z \quad (3)$$

$$\hat{A}_1(\tau) = U(\tau)A_1(\tau) = e^{-a\tau}, \quad \hat{A}_2(\tau) = U(\tau)A_2(\tau) = e^{-b\tau} \quad \hat{A}_3(\tau) = U(\tau)A_3(\tau) = e^{-c\tau} \quad (4)$$

where  $a \neq b \neq c$  are integration constants such that at least two are non-zero (otherwise space-time reduces to an LRS space-time). The commutators of the extra KV  $\mathbf{Y}$  with the standard KVs  $\xi_\mu$  are:

$$[\xi_1, \mathbf{Y}] = a\xi_1, [\xi_2, \mathbf{Y}] = b\xi_2, [\xi_3, \mathbf{Y}] = c\xi_3.$$

In conclusion we have the following result:

*All Bianchi I metrics which admit a CKV are ( $a \neq b \neq c$ ):*

$$ds^2 = -dt^2 + A_1^2(t) [dx^2 + e^{2(a-b)L(t)} dy^2 + e^{2(a-c)L(t)} dz^2] \quad (5)$$

where:

$$A_1(t) = \frac{1}{U(t)} e^{-a \int U(t)dt}. \quad (6)$$

The CKV is given by:

$$\mathbf{Y} = \frac{1}{U(t)} \partial_t + ax\partial_x + by\partial_y + cz\partial_z \quad (7)$$

and has conformal factor:

$$\phi(\mathbf{Y}) = a + \frac{1}{U(t)} [\ln |A_1(t)|]_{,t}. \quad (8)$$

In terms of the time coordinate  $\tau$  this metric is:

$$ds^2 = \frac{1}{U^2(\tau)} [-d\tau^2 + e^{-2a\tau} dx^2 + e^{-2b\tau} dy^2 + e^{-2c\tau} dz^2] \quad (9)$$

It is now easy to determine all Bianchi I space-times which admit (one) proper HVF. Indeed setting  $\phi(\mathbf{Y}) = \text{const.}$  ( $\neq 0$ ) and using (8) we find  $U(t) = \frac{1}{\phi t}$  from which it follows [19] (ignoring some unimportant integration constants):

**Proposition 1** *The only Bianchi I space-times which admit proper HVF are the Kasner-type space-times given by ( $a \neq b \neq c$ ):*

$$ds^2 = -dt^2 + t^{2\frac{\phi-a}{\phi}} dx^2 + t^{2\frac{\phi-b}{\phi}} dy^2 + t^{2\frac{\phi-c}{\phi}} dz^2. \quad (10)$$

The HVF is:

$$\mathbf{Y} = \phi t \partial_t + ax \partial_x + by \partial_y + cz \partial_z \quad (11)$$

and has homothetic factor  $\phi$ .

We remark that the same result can be recovered from the reduction of the proper RCs which will be determined in Section III.

## 2.2 Bianchi I space-times of Petrov type O

The necessary and sufficient condition for conformal flatness is the vanishing of the Weyl tensor  $C_{abcd}$ . By solving directly the equations  $C_{abcd} = 0$  it can be shown that there exist only two families of conformally flat Bianchi I metrics (we ignore the case of the Friedmann-Robertson-Walker space-time) given by:

$$ds_1^2 = A_3^2(\tau) ds_{RT}^2 \quad (12)$$

$$ds_2^2 = A_3^2(\tau) ds_{ART}^2. \quad (13)$$

The metrics  $ds_{RT}^2, ds_{ART}^2$  have been found previously by Rebouças-Tiomno [20] and Rebouças-Teixeira [21] respectively and are 1+3 (globally) decomposable space-times whose 3-spaces are spaces of constant curvature. In synchronous co-ordinates they are:

$$ds_{RT}^2 = dz^2 - d\tau^2 + \cos^2\left(\frac{\tau}{a}\right) dy^2 + \sin^2\left(\frac{\tau}{a}\right) dx^2 \quad (14)$$

$$ds_{ART}^2 = dz^2 - d\tau^2 + \cosh^2\left(\frac{\tau}{a}\right) dy^2 + \sinh^2\left(\frac{\tau}{a}\right) dx^2 \quad (15)$$

where  $d\tau = \frac{dt}{A_3(t)}$ . Each metric admits 15 CKVs which can be determined using standard techniques [22]. In concise notation these vectors are (we ignore the KVs  $\xi_\mu$  which constitute the  $G_3$ ) ( $k = 0, 1, 2, 3$  and  $\alpha = 0, 1, 2$ ):

**KVs**

$$\begin{aligned} \xi_{k+4} = & \left\{ s_\pm(y, a) \left[ \delta_k^0 c_+(x, a) + \delta_k^1 s_+(x, a) \right] + c_\pm(y, a) \left[ \delta_k^2 c_+(x, a) + \delta_k^3 s_+(x, a) \right] \right\} \partial_\tau + \\ & + \frac{s_\mp(\tau, a)}{c_\mp(\tau, a)} \left\{ c_\pm(y, a) \left[ \delta_k^0 c_+(x, a) + \delta_k^1 s_+(x, a) \right] \pm s_\pm(y, a) \left[ \delta_k^2 c_+(x, a) + \delta_k^3 s_+(x, a) \right] \right\} \partial_y - \\ & - \frac{c_\mp(\tau, a)}{s_\mp(\tau, a)} \left\{ s_\pm(y, a) \left[ \delta_k^0 s_+(x, a) + \delta_k^1 c_+(x, a) \right] + c_\pm(y, a) \left[ \delta_k^2 s_+(x, a) + \delta_k^3 c_+(x, a) \right] \right\} \partial_x \end{aligned} \quad (16)$$

**CKVs**

$$X_{(k)\alpha} = \pm a^2 B_{k,\alpha} \quad X_{(k)3} = \mp a^2 B_{k,3} \quad (17)$$

$$\phi(\mathbf{X}_{(k)}) = B_k \quad (18)$$

$$X_{(k+4)\alpha} = \pm a^2 \Gamma_{k,\alpha} \quad X_{(k+4)3} = \mp a^2 \Gamma_{k,3} \quad (19)$$

$$\phi(\mathbf{X}_{(k+4)}) = \Gamma_k \quad (20)$$

where:

$$B_k = c_{\mp}(\tau, a) \{c_{\pm}(y, a) [s_{\mp}(z, a), c_{\mp}(z, a)], s_{\pm}(y, a) [s_{\mp}(z, a), c_{\mp}(z, a)]\} \quad (21)$$

$$\Gamma_k = s_{\mp}(\tau, a) \{c_{+}(x, a) [s_{\mp}(z, a), c_{\mp}(z, a)], s_{+}(x, a) [s_{\mp}(z, a), c_{\mp}(z, a)]\}. \quad (22)$$

and the following conventions have been used:

1. The upper sign corresponds to RT space-time and the lower sign to ART space-time.
2. The functions  $s_{\mp}(w, a), c_{\mp}(w, a)$  are defined as follows:

$$(c_{+}(w, a), c_{-}(w, a)) = (\cosh(\frac{w}{a}), \cos(\frac{w}{a})) \quad (23)$$

$$(s_{+}(w, a), s_{-}(w, a)) = (\sinh(\frac{w}{a}), \sin(\frac{w}{a})) \quad (24)$$

The Bianchi I metrics (12), (13) have the same CKVs with conformal factors  $\psi(\boldsymbol{\xi}_{k+4}) = \boldsymbol{\xi}_{k+4}(\ln A_3)$  and  $\psi(\mathbf{X}_A) = \mathbf{X}_A(\ln A_3) + \phi(\mathbf{X}_A)$  ( $A = 1, 2, \dots, 8$ ). It follows that conformally flat Bianchi I space-times do not admit HVFs. Furthermore if we enforce them to admit an extra KV they reduce to the RT and the ART space-times which admit seven KVs.

### 3 Ricci Collineations

A RC  $\mathbf{X} = \mathbf{X}^a \partial_a$  is defined by the condition:

$$\mathcal{L}_{\mathbf{X}} R_{ab} = R_{ab,c} X^c + R_{ac} X_{,b}^c + R_{bc} X_{,a}^c = 0. \quad (25)$$

For RCs there do not exist theorems of equal power to the Theorem of Defrise-Carter and one has to solve directly the differential equations (25). However there do exist some general results available which are due to Hall and are summarised in the following statement [23]:

*If the Ricci tensor is of rank 4, at every point of the space-time manifold, then the smooth ( $C^2$  is enough) RCs form a Lie algebra of smooth vector fields whose dimension is  $\leq 10$  and  $\neq 9$ . This Lie algebra contains the proper RCs and their degeneracies.*

It has been pointed out by Hall and his co-workers that the assumption on the order of the Ricci tensor is important. Indeed let us assume that the order of the Ricci tensor of a Bianchi I space-time is 3. Then due to the fact that the Ricci tensor in Bianchi I space-times is diagonal one of its components must vanish, the  $R_{11} = 0$  say. This is equivalent to  $R_{ab} \xi_1^b = 0$  where  $\xi_1 = \partial_x$ . Consider the vector field  $\mathbf{X}_1 = f(x^a) \partial_x$  where  $f(x^a)$  is an arbitrary (but smooth)

function of its arguments. It is easy to show that  $\mathcal{L}_{\mathbf{X}_1} R_{ab} = 0$  so that  $\mathbf{X}_1$  is a Ricci collineation. Due to the arbitrariness of the function  $f(x^a)$  these Bianchi I space-times admit infinitely many smooth RCs a result that does not help us in any useful or significant way in their study.

In the following we consider smooth RCs and we assume that  $R_{ab}$  is non-degenerate ( $\det R_{ab} \neq 0$ ). Equation (25) gives the following set of 10 differential equations (no summation over the indices  $\mu, \nu, \rho = 1, 2, 3; \mu \neq \nu \neq \rho$ ):

$$[00] \quad R_{00,0} X^0 + 2R_{00} X_{,0}^0 = 0 \quad (26)$$

$$[0\mu] \quad R_{00} X_{,\mu}^0 + R_{\mu\mu} X_{,0}^\mu = 0 \quad (27)$$

$$[\mu\mu] \quad R_{\mu\mu,0} X^0 + 2R_{\mu\mu} X_{,\mu}^\mu = 0 \quad (28)$$

$$[\mu\nu] \quad R_{\mu\mu} X_{,\nu}^\mu + R_{\nu\nu} X_{,\mu}^\nu = 0 \quad (29)$$

where a comma denotes partial differentiation w.r.t. following index co-ordinate. For convenience we set  $R_{00} \equiv R_0, R_{\mu\mu} \equiv R_\mu$ . Equation (26) is solved immediately to give:

$$X^0 = \frac{m(x^\beta)}{\sqrt{|R_0|}} \quad (30)$$

where  $m(x^\beta)$  is a smooth function of the spatial co-ordinates. Using (30) we rewrite the remaining equations (27)-(29) in the form:

$$X_{,0}^\mu = -\epsilon_0 \frac{\sqrt{|R_0|}}{R_\mu} m_{,\mu} \quad (31)$$

$$X_{,\mu}^\mu = -\frac{(\ln |R_\mu|)_{,0}}{2\sqrt{|R_0|}} m \quad (32)$$

$$R_\mu X_{,\nu}^\mu + R_\nu X_{,\mu}^\nu = 0 \quad (33)$$

where  $\mu \neq \nu$  and  $\epsilon_0$  is the sign of the component  $R_0$ .

Differentiating equation (33) w.r.t.  $x^\rho$  we obtain:

$$R_\mu X_{,\nu\rho}^\mu + R_\nu X_{,\mu\rho}^\nu = 0. \quad (34)$$

Rewriting (33) for the indices  $\mu, \rho$ , differentiating w.r.t.  $x^\nu$  and subtracting from (34) we obtain:

$$R_\nu X_{,\mu\rho}^\nu - R_\rho X_{,\nu\mu}^\rho = 0. \quad (35)$$

Writing (33) for the indices  $\nu, \rho$ , differentiating w.r.t.  $x^\mu$  and adding to (35) ( $R_\mu \neq 0$ ) we get:

$$X_{,\mu\rho}^\nu = 0 \quad (\mu \neq \nu \neq \rho). \quad (36)$$

Differentiating (31) w.r.t.  $x^\mu$  and (32) w.r.t.  $x^0$  we find:

$$\frac{R_\mu}{\sqrt{|R_0|}} \left[ \frac{(\ln |R_\mu|)_{,0}}{2\sqrt{|R_0|}} \right]_{,0} = a_\mu \quad (37)$$

$$m_{,\mu\mu} = \epsilon_0 a_\mu m \quad (38)$$

where  $a_\mu$  are arbitrary constants.

Equations (31)-(33) and (36)-(38) constitute a set of differential equations in the variables  $(R_\mu, X^\mu, m)$ , which can be solved in terms of  $R_0$  and some integration constants. These constants are constrained by a set of algebraic equations involving the spatial components of the Ricci tensor and essentially determine the dimension of the algebra of the RCs.

The complete set of the solutions consists of five main cases which are summarized in TABLE I together with the corresponding algebraic constraints and the dimension of the resulting algebra.

To find the RCs we note that in all cases the component  $X^0$  is given by (30) and it is determined in terms of the function  $m(x^\alpha)$ . Concerning the spatial components these are obtained from the following formulas taking into consideration the fourth column of TABLE I.

#### Case A

$$X_I^\mu = c_\mu \left\{ -d \cdot x^\mu - x^\mu \sum_{\nu \neq \mu} D_\nu x^\nu - \frac{D_\mu}{2} \left[ (x^\mu)^2 - \sum_{\nu \neq \mu} \left( \frac{C_\mu}{C_\nu} \right) (x^\nu)^2 - \frac{\epsilon_0}{c_\mu^2 R_\mu} \right] \right\} + \sum_{\nu \neq \mu} B_\nu^\mu x^\nu \quad (39)$$

#### Case B ( $A = 2, 3$ )

$$X_{II}^A = c_A \left\{ -d \cdot x^A - x^A D_B x^B - \frac{D_A}{2} \left[ (x^A)^2 - \left( \frac{C_A}{C_B} \right) (x^B)^2 - \frac{\epsilon_0}{c_A^2 R_A} \right] \right\} + \Lambda_B^A x^B. \quad (40)$$

and  $X_{II}^1 = 0$ .

#### Case Ca

$$X_{III}^1 = -\frac{\epsilon_0 m_{,1}}{R_1} \int \sqrt{|R_0|} dt + b_2^1 x^2, \quad X_{III}^2 = -\frac{\epsilon_0 m_{,2}}{R_2} \int \sqrt{|R_0|} dt + b_1^2 x^1 \quad (41)$$

$$X_{III}^3 = -\frac{\epsilon_0 m_{,3}}{2a\sqrt{|R_0|}} (\ln |R_3|)_{,t} \quad (42)$$

#### Case Cb

$$X_{IV}^1 = b_2^1 x^2, \quad X_{IV}^2 = b_1^2 x^1 \quad (43)$$

$$X_{IV}^3 = \frac{\epsilon_0 D_3}{2c_3 R_3} - c_3 \left[ D_3 \frac{(x^3)^2}{2} + d \cdot x^3 \right] \quad (44)$$

#### Case D

$$X_V^\mu = \sum_{\nu \neq \mu} b_\nu^\mu x^\nu + f^\mu(x^0) \quad (45)$$

where:



$$f^\mu(x^0) = -\frac{D_\mu}{R_\mu} \int \sqrt{|R_0|} dx^0. \quad (46)$$

We collect the above results in the following:

**Proposition 2** *The proper smooth RCs in Bianchi I space-times can be considered in four sets depending on the constancy of the spatial Ricci tensor components. The first set (case A with  $R_{\mu,0} \neq 0$  for all  $\mu = 1, 2, 3$ ) contains three families of smooth RCs consisting of either one, two or seven RCs defined by the vector  $\mathbf{X}_I$  given by (39). The second set (case B with one  $R_{\mu,0} = 0$ ) consists of two families of one and four RCs defined by the vector  $\mathbf{X}_{II}$  given by (40). The third set (case C with two  $R_{\mu,0} = 0$ ) consists of five families with three, three, three, seven and three RCs are given by the vector fields  $\mathbf{X}_{III}, \mathbf{X}_{IV}$  defined in (41)-(42), (43)-(44). Finally the last set (case D all  $R_{\mu,0} = 0$ ) contains one family of seven RCs given by the vector field  $\mathbf{X}_V$  defined in (46). In each family of RCs the spatial Ricci tensor components are given in terms of the time component  $R_0$  and the coefficients of the vector fields are constrained with the spatial components of the Ricci tensor via algebraic conditions.*

We note that RCs do not fix the metric up to a set of constants as the CKVs (and the lower symmetries) do but instead they impose algebraic conditions on the metric functions. Thus in general one should expect many families of Bianchi I space-times admitting RCs.

## 4 Curvature Collineations

Curvature collineations are necessarily Ricci Collineations and one is possible to determine them from the results of the last section. The easiest way to do this would appear to use algebraic computing algorithms [24] and compute directly the Lie derivative of the curvature tensor for the RCs found in the last section. However this is not so obvious because although we know the RC we do not know the metric functions. Hence, in general, one expects to arrive at a system of differential equations among the metric functions  $A_\mu(t)$  whose solution will give the answer.

However a study of the relevant literature shows that there are enough general results which allow one to determine the CCs without solving any differential equations. The CCs have many of the pathologies of RCs. For example for any positive integer  $k$  there are CCs which are  $C^k$  but not  $C^{k+1}$ . Furthermore they may form an infinite dimensional vector space which is not a Lie algebra under the usual Lie bracket operation. However if one considers the  $C^\infty$  CCs only (loosing in that case the ones that are not smooth) then they do form a Lie algebra which is a subalgebra of the (smooth) RCs algebra [25, 26].

For the determination of CCs in Bianchi I space-times it is enough to use the following result from an early work of Hall [27]:

*If the curvature components are such that at every point of a space-time  $M$  the only solution of the equation  $R_{abcd}k^d = 0$  is  $k^d = 0$  then every CC on  $M$  is a HVF.*

Let us assume that in a Bianchi I space-time the equation  $R_{abcd}k^d = 0$  has a solution  $k^a \neq 0$ . Then it follows that equation  $R_{ab}k^a = 0$  admits a non-vanishing solution which is impossible because  $R_{ab}$  is non-degenerate. Hence according to the above statement all  $C^\infty$  CC in Bianchi I space-times are HVFs or, equivalently *there are no Bianchi I space-times (with non-degenerate Ricci tensor) which admit proper CCs.*

## 5 Kinematic Self Similarities

As it has been mentioned in the Introduction, Kinematic Self Similarities are not geometric symmetries (i.e. collineations). They are kinematic symmetries/constraints which involve the 4-velocity of the fluid (or in empty space-times a timelike unit vector field) defined by the conditions:

$$\mathcal{L}_X u_a = \alpha u_a \quad \mathcal{L}_X h_{ab} = 2\delta h_{ab} \quad (47)$$

where  $h_{ab} = g_{ab} + u_a u_b$  projects normally to  $u_a$  and  $\alpha, \delta$  are constants.

Kinematic self similarities have been studied by Sintès[28] who determined all LRS perfect fluid space-times which admit a KSS. Our aim in this Section is to determine the KSS of (proper) Bianchi I metrics without any restriction on the type of the fluid except that we assume that the fluid 4-velocity is orthogonal to the group orbits i.e.  $u^a = \delta_0^a$ . This assumption enforces the commutator of a KSS  $\mathbf{X}$  with the three KVs  $\xi_\mu$  to be a KV[29]. Hence we write:

$$[\xi_\mu, \mathbf{X}] = X_{,\mu}^a \partial_a = a_\mu^\nu \xi_\nu \quad (48)$$

where  $a_\mu^\nu$  are constants. Integrating we find:

$$X^0 = X^0(t) \quad \text{and} \quad X^\mu = a_\nu^\mu x^\nu + f^\mu(t) \quad (49)$$

where  $f^\mu(t)$  are arbitrary smooth functions of their argument. The first of equations (47) gives:

$$X^0 = \alpha t + \beta$$

where  $\beta$  is an integration constant and (without loss of generality)  $f^\mu(t) = \text{const.} = 0$ . The second equation of (47) gives the following conditions among the metric functions:

$$a_\nu^\mu (A_\mu)^2 + a_\mu^\nu (A_\nu)^2 = 0 \quad (50)$$

$$a_\mu^\mu + (\alpha t + \beta) \frac{d(\ln A_\mu)}{dt} = \delta \quad (51)$$

where  $\mu \neq \nu$  and a dot denotes differentiation w.r.t.  $t$ . Equation (50) means that, in order to avoid the plane symmetric case, we must take  $a_\nu^\mu = 0$  for  $\mu \neq \nu$ . Therefore we have the following conclusion ( $a_\mu^\mu \equiv a_\mu$ ):

**Proposition 3** *The Bianchi I space-times whose metric functions satisfy the relation ( $a_\mu \in \mathbb{R}$ ):*

$$a_\mu + (\alpha t + \beta) \frac{d(\ln A_\mu)}{dt} = \delta \quad (52)$$

*admit the proper KSS:*

$$\mathbf{X} = (\alpha t + \beta) \partial_t + a_1 x \partial_x + a_2 y \partial_y + a_3 z \partial_z. \quad (53)$$

In view of equation (52) we have two distinct cases to consider, namely  $\alpha = 0$  (type zero) and  $\alpha \neq 0$ .

Case  $\alpha = 0$  ( $\delta \neq 0$ ).

For  $\beta = 0$  equation (52) is trivially satisfied i.e. *all Bianchi I space-times admit the (zeroth kind) KSS*:

$$\mathbf{Z} = \delta(x\partial_x + y\partial_y + z\partial_z). \quad (54)$$

For  $\beta \neq 0$  we have:

$$A_\mu(t) = e^{\frac{\delta - a_\mu}{\beta}t}. \quad (55)$$

It is easy to show that in this case space-time admits the KV:

$$\mathbf{Y} = \partial_t + \frac{a_1 - \delta}{\beta}x\partial_x + \frac{a_2 - \delta}{\beta}y\partial_y + \frac{a_3 - \delta}{\beta}z\partial_z \quad (56)$$

and becomes homogeneous. (The KV (56) can also be found from the reduction of the results of Section II).

Case  $\alpha \neq 0$

In this case the solution of (52) is:

$$A_\mu(t) = (\alpha t + \beta)^{\frac{\delta - a_\mu}{\alpha}} \quad (57)$$

and leads to the conclusion that:

*The only Bianchi I space-times with co-moving fluid which admit a KSS of the second kind are the Kasner type space-times.*

We note that the Bianchi I space-times with metric functions given by equation (57) also admit the HVF:

$$\alpha\partial_t + (\alpha + a_1 - \delta)x\partial_x + (\alpha + a_2 - \delta)y\partial_y + (\alpha + a_3 - \delta)z\partial_z \quad (58)$$

with homothetic factor  $\alpha$  (see Section II).

## 6 Discussion

Working with purely geometric methods we have succeeded to determine all (proper and diagonal) Bianchi I space-times which admit certain (and the most important) collineations. In many cases the explicit form of the metrics has been given (CKVs, HVFs, KSS), in others the metrics are defined up to a set of conditions on the metric functions (RCs) and finally it has been shown that there are not Bianchi I space-times which admit CCs and ACVs. In all cases the collineation vector has been determined (whenever it exists).

In order to establish the physical significance of these general geometrical results we address the following questions for each type of collineation:

1. *Are there any Bianchi I metrics among the ones selected by one of the collineations considered, which satisfy the energy conditions, so that they can be used as potential space-time metrics?*

2. *If there are, are the known Bianchi I solutions among these solutions?*

In the following we take the cosmological constant  $\Lambda = 0$ .

## 6.1 The case of CKVs

Of interest is only the non-conformally flat metrics (6) (or (7)) which admit the CKV (9) with conformal factor (8).

One general result is that the CKV is inheriting, that is, the fluid flow lines are preserved under Lie transport along the CKV[30].

Concerning the dynamical results we consider three main cases: perfect fluids, non null Einstein-Maxwell solutions and imperfect fluid solutions (the case of null Einstein-Maxwell field[31] is excluded because the Segré type of Bianchi I space-times is  $[1, 111]$  or degeneracies of this type).

### Perfect Fluid Solutions

In this case the Segré type of space-time is  $[1, (111)]$  and all spatial eigenvalues  $\lambda_\mu$  are equal. Moreover  $\lambda_\mu = G^\mu_\mu$  where  $G_{ab}$  is the Einstein tensor. Considering  $\lambda_2 = \lambda_3$  and using (8) we find  $U = \frac{1}{Bt}$  where  $B = \frac{a+b+c}{2}$ . Finally replacing  $U(t)$  in the line element (6) we find that the resulting space-time is a Kasner type space-time and the CKV reduces to a HVF in agreement with the general result that *orthogonal spatially homogeneous perfect fluid space-times do not admit any inheriting proper CKV*. [30] Most of the known Bianchi I solutions concern perfect fluid solutions[32]. None of these solutions admit a CKV.

### Einstein Maxwell solutions

For these fields the Segré type of the Einstein tensor is  $[(1,1)(11)]$  hence  $\lambda_2 = \lambda_3$  and  $\lambda_0 = \lambda_1$ . The first equality implies again that the metric reduces to a Kasner type metric and in fact to a vacuum solution (because if we force a Kasner type metric to represent a (necessarily non null) Einstein-Maxwell field it reduces to a vacuum solution).

**Proposition 4** *There do not exist Bianchi I (non-null) Einstein-Maxwell space-times which admit a CKV or a HVF.*

The two Bianchi I solutions with electromagnetic field found by Datta [33] and by Rosen [34] do not admit a CKV or a HVF.

### Anisotropic fluid solutions

The above results indicate that the Bianchi I metrics (6) can represent only anisotropic fluid space-times. Recently anisotropic fluid Bianchi I cosmological models have been investigated extensively using a dynamical system approach and the truncated Israel-Stewart theory of irreversible thermodynamics. It has been found that in these models, anisotropic stress leads to models which violate the weak energy condition, thus they are unphysical or they lead to the creation of a periodic orbit [35, 36, 37, 38, 39, 40].

In order to find one such solution which will be physically viable we consider the following two restrictions:

1.  $c = 0$  and
2. The algebraic type of matter (equivalently Einstein) tensor is  $[(1,1)11]$ .

Setting  $\lambda_0 = \lambda_3$  we obtain the condition:

$$2(\ln M)_{,\tau\tau} - 2[(\ln M)_{,\tau}]^2 + b^2 + a^2 = 0 \quad (59)$$

whose solution is:

$$M(\tau) = \frac{1}{\sinh k\tau}, \frac{1}{\cosh k\tau} \quad (60)$$

where  $k^2 = \frac{a^2+b^2}{2}$ . We keep the solution  $M(\tau) = \frac{1}{\sinh k\tau}$  because the other violates all energy conditions. This gives  $U(\tau) = \sinh k\tau$  or  $U(t) = \sinh^{-1} kt$  and finally we obtain the metric:

$$ds^2 = -dt^2 + \sinh^{2\frac{k-a}{k}} \frac{kt}{2} \cosh^{2\frac{k+a}{k}} \frac{kt}{2} dx^2 + \sinh^{2\frac{k-b}{k}} \frac{kt}{2} \cosh^{2\frac{k+b}{k}} \frac{kt}{2} dy^2 + \sinh^2 ktdz^2. \quad (61)$$

This new Bianchi I space-time describes a viscous fluid and satisfies the weak and the dominant energy conditions (a description of these energy conditions in terms of the eigenvalues of the stress-energy tensor is given in the Appendix) provided  $ab > 0$  and  $ka < 0$ . The strong energy condition is violated. It also admits the proper CKV  $\mathbf{X} = \sinh kt \partial_t + ax \partial_x + by \partial_y$  with conformal factor  $\phi(\mathbf{X}) = k \cosh kt$ .

To study the physics of the new solution we consider the stress-energy tensor  $T_{ab}$  and using the standard Eckart theory we write:

$$T_{ab} = \mu u_a u_b + (\bar{p} - \zeta \theta) h_{ab} - 2\eta \sigma_{ab} \quad (62)$$

where  $\zeta, \eta \geq 0$  are the bulk and the shear viscosity coefficients and  $\bar{p}$  is the isotropic pressure in the absence of dissipate processes i.e.  $\zeta = 0$  (equilibrium state). It is easy to show that vanishing of  $\zeta$  together with a linear barotropic equation of state  $\bar{p} = (\gamma - 1)\mu$  (where  $\gamma \in [1, 2]$ ) lead to the condition  $a = -b$ , which violates the weak energy condition ( $ab < 0$ ). Thus we restrict our study to the case where  $\zeta \neq 0$  which is of cosmological interest. For example inflation driven by a viscous fluid necessarily involves bulk viscous stress[41]. Moreover cosmological models which include viscosity can be used in an attempt to interpret the observed highly isotropic matter distribution[43]. In fact it has been shown that viscosity plays a significant role in the isotropisation of the cosmological models.[44]

Using standard methods we find for the kinematic and the dynamic variables of the model:

$$\begin{aligned} \mu &= \frac{3k^2 \cosh^2 kt - 2k(b+a) \cosh kt + ab}{\sinh^2 kt} \\ \zeta &= \frac{1}{\theta} \left[ \mu + \bar{p} + \frac{2k(a+b) \cosh kt - 2(k^2 + ab)}{3 \sinh^2 kt} \right] \\ \theta &= \frac{3k \cosh kt - (b+a)}{\sinh kt} \end{aligned} \quad (63)$$

$$\begin{aligned} \sigma_{11} &= \frac{(b-2a)}{6} \sinh^{\frac{k-2a}{k}} \frac{kt}{2} \cosh^{\frac{k+2a}{k}} \frac{kt}{2} \\ \sigma_{22} &= \frac{(a-2b)}{6} \sinh^{\frac{k-2b}{k}} \frac{kt}{2} \cosh^{\frac{k+2b}{k}} \frac{kt}{2} \\ \sigma_{33} &= \frac{(a+b)}{6} \sinh kt \end{aligned} \quad (64)$$

$$\sigma^2 = \frac{a^2 - ab + b^2}{3 \sinh^2 kt} \quad (65)$$

where  $2\sigma^2 = \sigma_{ab}\sigma^{ab}$  and:

$$\eta = \frac{a+b}{2 \sinh kt} - k \coth kt. \quad (66)$$

The explicit computation of the bulk viscosity  $\zeta$  requires the adoption of a specific equation of state in order to guarantee that  $\zeta$  is positive definite. However the simple choice  $a+b > 0$  and  $k < 0$  ensures that  $\zeta \geq 0$  (since  $\theta < 0$ ) and  $\eta > 0$  provided that  $\bar{p} \leq p_{eff} \equiv \bar{p} - \zeta\theta$ .

Concerning the asymptotic behavior of the model we have  $\lim_{t \rightarrow \infty} \sigma = 0$  provided that  $k < 0$  and the model isotropizes at late times. Furthermore  $\lim_{t \rightarrow \infty} \theta = -3k$ ,  $\lim_{t \rightarrow \infty} R = 12k^2$ ,  $(\mu + p_{eff})_{t \rightarrow \infty} = 0$  hence the model corresponds to the flat FRW space-time i.e. the de Sitter universe. This also follows directly from the metric (61) if we consider the limit  $t \rightarrow \infty$ . In this limit the CKV  $\mathbf{X}$  degenerates to a KV. In addition using (63) it can be shown that there is a cosmological singularity of Kasner type at a *finite* time in the past i.e. there is a  $t = t_0$  at which the energy density vanishes, whereas the initial singularity occurs at  $t = 0$ .

## 6.2 The case of Ricci Collineations

There are four families of Bianchi I space-times which admit RCs and furthermore the metric in these families is fixed only up to a set of algebraic conditions. Due to this generality we are obliged to consider again special cases and the best choice is perfect fluid solutions.

The algebraic type of the Ricci tensor for a perfect fluid is  $[1, (111)]$  which implies the condition:

$$\frac{R_{11}}{A_1^2} = \frac{R_{22}}{A_2^2} = \frac{R_{33}}{A_3^2}. \quad (67)$$

This immediately excludes the last three families (B,Ca,Cb,D) of TABLE I and we are left with family A only. From the first column of TABLE I we read  $R_\mu = C_\mu e^{\int 2c_\mu \sqrt{|R_0|} dt}$  hence (67) imply:

$$\frac{C_1}{C_2} e^{2(c_1-c_2) \int \sqrt{|R_0|} dt} = \frac{A_1^2}{A_2^2} \quad \text{and} \quad \frac{C_1}{C_3} e^{2(c_1-c_3) \int \sqrt{|R_0|} dt} = \frac{A_1^2}{A_3^2}. \quad (68)$$

In order to avoid the FRW metric ( $A_1 \propto A_2 \propto A_3$ ) and the plane symmetric metric (e.g.  $A_1 \propto A_2$ ) we demand  $c_1 \neq c_2 \neq c_3$ . Then from the third column of TABLE I we have that only case  $A_2$  survives and furthermore  $D_\mu = b_\nu^\mu = 0$ . Setting the constant  $d = 1$  we find from (39):

$$\mathbf{X} = \frac{1}{\sqrt{R_{00}}} \partial_0 - c_1 x^1 \partial_1 - c_2 x^2 \partial_2 - c_3 x^3 \partial_3. \quad (69)$$

We conclude that all perfect fluid Bianchi I space-times which satisfy (68) admit a RC of the form (69).

As far as we aware all existing perfect fluid solutions in Bianchi I space-times concern perfect fluids with a linear barotropic equation of state  $p = (\gamma - 1)\mu$  ( $\gamma \in [1, 2]$ ). In order to compare

these solutions with the above perfect fluid solutions we assume that the later also satisfy a linear barotropic equation of state. Then from the field equations we obtain:

$$R_{ab} = \mu \left[ \frac{(2-\gamma)}{2} g_{ab} + \gamma u_a u_b \right]. \quad (70)$$

(The value  $\gamma = 2$  is excluded because then the Ricci tensor becomes degenerated). The 00 conservation equation gives:

$$\mu_{,t} + \gamma \mu \theta = 0 \quad (71)$$

where  $\theta = u^a_{;a}$  is the expansion of the fluid. Introducing the scale factor or "mean radius"  $S^3 = A_1 A_2 A_3$  we find  $\theta = (\ln S^3)_{,t}$  and the solution of equation (71) is:

$$\mu = \frac{M}{S^{3\gamma}} \quad (72)$$

where  $M = \text{constant}$ . Using (68), (70) and (72) we find:

$$\mu = \frac{\text{const.}}{t^2}. \quad (73)$$

Combining this with (67) and (68) we find  $A_1 \propto t^p, A_2 \propto t^q, A_3 \propto t^r$  i.e. the resulting Bianchi space-time is a Kasner type space-time which is a contradiction because a Kasner type space-time with a perfect fluid leads to stiff matter[32] i.e.  $\gamma = 2$ . Thus we have proved:

**Proposition 5** *All perfect fluid Bianchi I space-times whose metric functions satisfy condition (68) admit a RC of the form (69). In addition perfect fluid Bianchi I space-times with linear barotropic equation of state do not admit proper RCs.*

Finally it should be pointed that although spatially homogeneous perfect fluid space-times must satisfy a barotropic equation of state  $p = p(\mu)$  this equation need not necessarily be linear. However it was proved recently that the asymptotic behaviour of such models is similar to the case of a linear barotropic equation of state [42].

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## Appendix

It is well known that Bianchi I space-times have zero heat flux and the 4-velocity  $u^a$  is an eigenvector of the energy momentum tensor. Consequently the only possible algebraic Segré type of their energy momentum tensor is [1, 111] and its degeneracies [45, 46, 47]. Furthermore, for this type of energy momentum tensors it has been shown that the energy conditions take the following form [48, 49]:

Weak energy condition

$$-\lambda_0 \geq 0 \text{ and } -\lambda_0 + \lambda_\alpha \geq 0 \quad (A1)$$

Dominant energy condition

$$-\lambda_0 \geq 0 \text{ and } |\lambda_\alpha| \leq -\lambda_0 \quad (\text{A2})$$

Strong energy condition

$$-\lambda_0 \geq 0 \text{ and } -\lambda_0 + \sum_{\alpha} \lambda_{\alpha} \geq 0 \quad (\text{A3})$$

where  $\lambda_0$  is the eigenvalue of the timelike eigenvector and  $\lambda_{\alpha}$  are the eigenvalues of the spacelike eigenvectors.

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- A Kasner type space-time is a spacetime metric of the form  $ds^2 = -dt^2 + t^p dx^2 + t^q dy^2 + t^r dz^2$  where the parametrs  $p, q, r$  are functions of the space coordinates  $x, y, z$  and they are not restricted by any relation.
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**TABLE I:** The table contains the complete set of solutions of the Ricci Collineation equations. The solutions are specified in terms of  $R_0$  and some integration constants together with the algebraic constraints (if any) which  $R_\mu$  must satisfy. The second column contains the spatial components of the Ricci tensor, the third the function  $m(x^\alpha)$ , the fourth column the constraints on the integration constants and the last column the number of proper RCs. The indices  $\mu, \nu, \rho = 1, 2, 3$ ,  $\mu \neq \nu \neq \rho$ ,  $A, B = 2, 3$ ,  $A \neq B$  and there is no summation over repeated indices. The functions  $c_\varepsilon(x^3, a)$ ,  $s_\varepsilon(x^3, a)$  are given in (23) and (24).

Case	$R_\mu$	$m(x^\alpha)$	Constraints	No of proper RCs
<b>A<sub>1</sub></b>	$R_\mu B_\nu^\mu + R_\nu B_\mu^\nu = 0$	0	none	3
<b>A<sub>2</sub></b>	$R_\mu = C_\mu e^{\int 2c_\mu \sqrt{ R_0 } dx^0}$	$d$	$c_\mu \neq c_\nu$ $D_\mu = B_\mu^\nu = 0$	1
<b>A<sub>3</sub></b>	$R_\mu = C_\mu e^{\int 2c_\mu \sqrt{ R_0 } dx^0}$ $R_\rho B_\nu^\rho + R_\nu B_\rho^\nu = 0$	$d$	$c_\rho = c_\nu \neq c_\mu$ $D_\mu = B_\mu^\nu = 0$	2
<b>A<sub>4</sub></b>	$R_\mu = C_\mu e^{\int 2c_\mu \sqrt{ R_0 } dx^0}$ $R_\rho B_\nu^\rho + R_\nu B_\rho^\nu = 0$	$\sum_{\mu=1,2,3} D_\mu x^\mu + d$	$c_\mu = c_\nu$	7
<b>B<sub>1</sub></b>	$R_{1,0} = 0$ $R_A = C_A e^{\int 2c_A \sqrt{ R_0 } dx^0}$	$d$	$c_A \neq c_B$ $D_A = \Lambda_A^B = 0$	1
<b>B<sub>2</sub></b>	$R_{1,0} = 0$ $R_A = C_A e^{\int 2c_A \sqrt{ R_0 } dx^0}$ $R_A \Lambda_B^A + R_B \Lambda_A^B = 0$	$\sum_{A=2,3} D_A x^A + d$	$c_A = c_B$	4
<b>Ca<sub>1</sub></b>	$R_{1,0} = R_{2,0} = 0$ $R_1 b_2^1 + R_2 b_1^2 = 0$ $R_3 = \epsilon_3 C_3^2 \cosh^2 \left[ \frac{(\epsilon_3 a)^{1/2} \int \sqrt{ R_0 } dt}{C_3} \right]$	$\beta_1 s_\varepsilon \left( x^3, \frac{1}{\sqrt{ \epsilon_0 a }} \right) +$ $\beta_2 c_\varepsilon \left( x^3, \frac{1}{\sqrt{ \epsilon_0 a }} \right)$	$\epsilon_3 a > 0$ $\varepsilon = \text{sign}(\epsilon_0 a)$	3
<b>Ca<sub>2</sub></b>	$R_1, R_2$ same as <b>Ca<sub>1</sub></b> $R_3 = \epsilon_3 C_3^2 \sinh^2 \left[ \frac{ \epsilon_3 a ^{1/2} \int \sqrt{ R_0 } dt}{C_3} \right]$	same as <b>Ca<sub>1</sub></b>	$\epsilon_3 a < 0$	3
<b>Ca<sub>3</sub></b>	$R_1, R_2$ same as <b>Ca<sub>1</sub></b> $R_3 = \epsilon_3 C_3^2 \cos^2 \left[ \frac{ \epsilon_3 a ^{1/2} \int \sqrt{ R_0 } dt}{C_3} \right]$	same as <b>Ca<sub>1</sub></b>	$\epsilon_3 a < 0$	3
<b>Ca<sub>4</sub></b>	$R_1, R_2$ same as <b>Ca<sub>1</sub></b> $R_3 = -\epsilon_3 a \left( \int \sqrt{ R_0 } dt \right)^2$	$\sum_\alpha m_\alpha (x^3) x^\alpha$ $m_\alpha = \gamma_\alpha^1 s_\varepsilon \sqrt{ \epsilon_0 a } x^3 +$ $+ \gamma_\alpha^2 c_\varepsilon \sqrt{ \epsilon_0 a } x^3$	$\varepsilon = \text{sign}(\epsilon_0 a)$	7
<b>Cb</b>	$R_1, R_2$ same as <b>Ca<sub>1</sub></b> $R_3 = C_3 e^{\int 2c_3 \sqrt{ R_0 } dt}$	$D_3 x^3 + d$	$c_3 \neq 0$	3
<b>D</b>	$R_{1,0} = R_{2,0} = R_{3,0} = 0$ $R_\mu B_\nu^\mu + R_\nu B_\mu^\nu = 0$	$\sum_{\mu=1,2,3} D_\mu x^\mu + d$	none	7